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Solution of Cubic-Quintic Duffing Oscillators using Harmonic Balance Method

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ABSTRACT

In this study, Harmonic Balance Method (HBM) is applied to determine approximate analytic solutions of strongly nonlinear Duffing oscillators with cubic-quintic nonlinear restoring force. Mainly, a set of nonlinear algebraic equations is solved in this method. The new method avoids the necessity of numerically solving sets of algebraic equations with very complex nonlinearities. Numerical comparisons between the HBM and the exact solutions reveal that the HBM is a promising tool for strongly nonlinear oscillator's problems.

Keywords: Approximate frequency, Harmonic balance method, Cubic-quintic Duffing oscillator, Analytical solutions.

1. Introduction

Basically in different fields of science and engineering there are few issues occurring linear whereas a great number of problems result in the nonlinear systems. Nonlinear oscillations are important fact in physical science, mechanical structures and other engineering problems. The methods of solutions of linear differential equations are comparatively easy and well established. On the contrary, the techniques of solutions of nonlinear differential equations (NDEs) are less available and, in general, linear approximations are frequently used. With the discovery of numerous phenomena of self- excitation of a strongly nonlinear cubic-quintic Duffing oscillator and in many cases of nonlinear mechanical vibrations of special types, the methods of small oscillations become inadequate for their analytical treatment.

In recent year, nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of approximate solutions of NDEs play a crucial role to understand the internal mechanism of nonlinear phenomena. Advanced nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving differential equations, dynamical systems and related areas. Presently, both mathematicians and physicists have made significant improvement in finding a new mathematical tool related to NDEs and dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. They establish many effective and powerful methods to handle the NDEs.

The study of given nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by NDEs. Moreover, obtaining exact solutions for these problems has many difficulties. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. To overcoming the shortcomings, many new analytical methods have proposed these days. One of the widely used techniques is perturbation and asymptotic method (Chowdhury, 2012, 2013, Cveticanin, 2010, Elmas, 2014, Gupta, 2014, Nayfeh, 1973, Nazari-Golshan, 2013, Sedighi and Attarzadeh, 2013), whereby the solution is expanded in powers of a small parameter. Perturbation method gives deviated results for strongly nonlinear oscillator. However, for the nonlinear conservative systems, generalizations of some of the standard perturbation techniques overcome this limitation. In particular, generalization of modified differential transforms method and He's homotopy perturbation method yield desired results even for strongly nonlinear oscillators (Belendez, 2009b, Nourazar, 2013). Several scientists also used various other powerful analytical methods in the field of approximate solutions especially for strongly nonlinear oscillators like Max-Min Approach Method (MMAM), Parameter Expansion Method (PEM), Variational Iteration Method (VIM) and Amplitude Frequency Formulation (AFF) for solving strongly NDEs (Azimi, 2012, Baghani, 2012, Ganji, 2012b, He, 2008). Some researchers already found the approximation solutions for strongly nonlinear cubic-quintic Duffing oscillators using He's Energy balance Method, Iteration Perturbation Method and Newton-harmonic balancing method (Ganji, 2012a, 2009, Lai, 2008). In these types of methods, with obtaining the motion frequency and having the initial conditions the result achieved. Most of the methods which discussed above the simplification procedure are tremendously difficult task especially for obtaining higher order approximations.

Harmonic balance method is another method for solving strongly NDEs (Hu, 2006, Mickens, 1996, 1984, 1986). Afterwards some researchers modified HBM for solving NDEs (Belendez, 2009a, Leung, 2012, Xiao, 2013). Generally, a set of difficult nonlinear algebraic equations are appear when harmonic balance method is formulated. But in classical harmonic balance method and some modification of harmonic balance method there is no clear idea for solving these complicated nonlinear higher order algebraic equations. To overcome this limitation, we have presented an analytical technique based on HBM for solving strongly nonlinear conservative systems (García-Saldana, 2013, Hosen, 2013a,b, 2014, 2012, Karkar, 2014, Rahman, Kong). The higher order approximations (mainly third-order approximation) have been obtained for strongly nonlinear cubic-quintic Duffing oscillators. Comparison of the approximate frequencies obtained in this article with its exact frequencies which shows that this method is effective and convenient for solving these analytical results.

2. Solution procedure

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x) \quad \text{and initial condition} \quad [x(0) = a_0, \dot{x}(0) = 0] \tag{1}$$

where $f(x)$ is a nonlinear function such that $f(-x) = -f(x)$, $\omega_0 \geq 0$ and ε is a constant. Consider a periodic solution of Eq. (1) is in the form

$$x = a_0(\rho cos(\omega t) + u cos(3\omega t) + v cos(5\omega t) + w cos(7\omega t) + z cos(9\omega t) + \cdots) \tag{2}$$

where a_0, ρ and ω are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, the solution of the form as Eq. (2) easily satisfies the initial condition (1). Substituting Eq. (2) into Eq. (1) and expanding $f(x)$ in a Fourier cosine series, it becomes to an algebraic identity

$$\begin{aligned} a_0[(\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)cos(3\omega t) + \cdots] = \\ -\varepsilon[F_1(a_0, u, \cdots)cos(\omega t) + F_3(a_0, u, \cdots)cos(3\omega t) + \cdots] \end{aligned} \tag{3}$$

where $F_i(a_0, u, v, \cdots)$, $i = (2n+1)$, $n = 0, 1, 2, \cdots$ is a nonlinear algebraic mathematical expression. By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

With help of the first equation, ω^2 is eliminated from all the remaining equations of Eq. (4). Thus Eq. (4) takes the following form

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1,\ 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1),\ 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \cdots \tag{5}$$

Substitution $\rho = 1 - u - v - \cdots$, and simplification, second-, third- equations of Eq. (5) take the following form

$$u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \cdots, \tag{6}$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of ω_0, ε and a_0 there exists a parameter $\lambda_0(\omega_0, \varepsilon, a_0) \ll 1$, such that $u, v, \cdots$ are expandable in the following power series in terms of λ_0 as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \cdots, \tag{7}$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. (2).

3. General Definition of Cubic-Quintic Duffing Oscillators

A cubic-quintic Duffing oscillator has the general form of

$$\ddot{x} + f(x) = 0, \tag{8}$$

with initial conditions of

$$x(0) = a_0, \quad \dot{x}(0) = 0, \tag{9}$$

where $f(x) = \alpha x + \beta x^3 + \gamma x^5$. x and t are generalized dimensionless displacement and time variable respectively and x is the function of t.
The exact frequency ω_{ex} is calculated by imposing the mentioned initial conditions is stated in (Lai, 2008) as

$$\omega_{ex}(a_0) = \frac{\pi k_1}{2\int_0^{\pi/2}(1 + k_2 \sin^2 t + k_3 \sin^4 t)^{-1/2} dt}, \tag{10}$$

where

$$k_1 = \sqrt{\alpha + \frac{\beta a_0^2}{2} + \frac{\gamma a_0^4}{4}},$$

$$k_2 = \frac{3\beta a_0^2 + 2\gamma a_0^4}{6\alpha + 3\beta a_0^2 + 2\gamma a_0^4},$$

$$k_3 = \frac{2\gamma a_0^4}{6\alpha + 3\beta a_0^2 + 2\gamma a_0^4}.$$

4. Solution procedure of Cubic-Quintic Duffing oscillators

The complete solution procedure of cubic-quintic duffing oscillator (Eq. (8)) with $\alpha = \beta = \gamma = 1$ is presented by HBM. Furthermore, the comparison between the frequencies obtained by HBM and the exact ones are shown in Table 1. This table corresponds to small as well as large amplitude of mentioned oscillator which obtained by HMB as described in the following example.
Rewrite the general form of cubic-quintic Duffing oscillator in Eq. (8), with $\alpha = \beta = \gamma = 1$ as

$$\ddot{x} + x + x^3 + x^5 = 0. \tag{11}$$

Herein we have to determine second- and third-order approximations of the frequency for the cubic-quintic Duffing oscillator.
Let us consider a two-term solution, i.e., $x = a_0(\rho\cos(\omega_2 t) + u\cos(3\omega_2 t))$ for

the Eq. (11). Substituting this solution along with $\rho = 1-u$ into the Eq. (11), Eq. (3) becomes

$$\begin{aligned}(1-u)\omega_2^2\cos(\omega_2 t) + 9u\omega_2^2\cos(3\omega_2 t) = (1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 \\ -25a_0^4u/16 + 9a_0^2u^2/4 + \cdots)\cos(\omega_2 t) + (a_0^2/4 + 5a_0^4/16 + u + 3a_0^2u/4 \\ -9a_0^2u^2/4 + \cdots)\cos(3\omega_2 t) + HOH,\end{aligned} \tag{12}$$

where HOH stands for higher order harmonics. Now comparing the coefficients of equal harmonics, the following equations are obtained

$$\begin{aligned}(1-u)\omega_2^2 &= 1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 + \\ &\quad 9a_0^2u^2/4 + 15a_0^4u^2/4 + \cdots \\ 9u\omega_2^2 &= a_0^2/4 + 5a_0^4/16 + u + 3a_0^2u/4 + 5a_0^4u/16 - \\ &\quad 9a_0^2u^2/4 - 5a_0^4u^2/2 + \cdots\end{aligned} \tag{13}$$

From the first equation of Eq. (13), it becomes

$$\begin{aligned}\omega_2^2 = (1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2u/2 - 25a_0^4u/16 + 9a_0^2u^2/4 + \\ 15a_0^4u^2/4 + \cdots)/(1-u)\end{aligned} \tag{14}$$

by elimination of ω_2^2 from the second equations of Eq. (13) with the help of Eq. (14) and simplification, the following nonlinear algebraic equation of u is found

$$\begin{aligned}-a_0^2/4 - 5a_0^4/16 + 8u + 25a_0^2u/4 + 45a_0^4u/8 - 8u^2 - \\ 21a_0^2u^2/2 - 45a_0^4u^2/4 + 16a_0^2u^3 + 25a_0^4u^3 - \\ 23a_0^2u^4/2 - 675a_0^4u^4/16 + 355a_0^4u^5/8 - 85a_0^4u^6/4 = 0\end{aligned} \tag{15}$$

The Eq. (15) can be written as

$$u = \lambda_0(4 + 5a_0^2 + 128u^2/a_0^2 + 168u^2 - 256u^3 + 184u^4 - 710a_0^2u^5 + \cdots), \tag{16}$$

where $\lambda_0 = a_0^2/(128 + 100a_0^2 + 90a_0^4)$. The power series solution of Eq. (16) in terms of λ_0 is obtained as

$$\begin{aligned}u = (4 + 5a_0^2)\lambda_0 + (7808 + 2048/a_0^2 + 12800a_0^2 + \\ 11400a_0^4 + 4500a_0^6)\lambda_0^3 + \cdots .\end{aligned} \tag{17}$$

Substituting the value of u from Eq.(17) into the Eq. (14) and then simplification the second order approximate angular frequency is appeared as the following

$$\begin{aligned}\omega_2 = \big[(16 - 36a_0^2 - 110a_0^4 - 75a_0^6)\lambda_0/16 + \\ (384a_0^2 + 1680a_0^4 + 2400a_0^6 + 1125a_0^8)\lambda_0^2/16 + \cdots\big]^{\frac{1}{2}}\end{aligned} \tag{18}$$

Thus a two-term solution (i.e. second approximation) of Eq. (11) is

$$x = a_0 \cos(\omega_2 t) + a_0 u(\cos(3\omega_2 t) - \cos(\omega_2 t)), \tag{19}$$

where u and ω_2 are given respectively by Eqs. (17) and (18).
In a similar way, the method can be used to determine higher-order approximations. In this article, a third approximate solution is,

$$x = a_0 \cos(\omega_3 t) + a_0 u(\cos(3\omega_3 t) - \cos(\omega_3 t)) + a_0 v(\cos(5\omega_3 t) - \cos(\omega_3 t)) \tag{20}$$

Substituting Eq. (20) into the Eq. (11) and equating the coefficients of same harmonic terms $\cos(\omega_3 t)$, $\cos(3\omega_3 t)$ and $\cos(5\omega_3 t)$ the related equations are

$$\begin{aligned}
(1-u-v)\omega_3^2 = 1 &+ 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2 u/2 - 25a_0^4 u/16 + \\
&9a_0^2 u^2/4 + 15a_0^4 u^2/4 - 3a_0^2 u^3/2 - 25a_0^4 u^3/4 + \\
&25a_0^4 u^4/4 - 45a_0^4 u^5/16 - v - 9a_0^2 v/4 - \\
&45a_0^4 v/16 + 9a_0^2 uv/2 + 10a_0^4 uv - \\
&3a_0^2 u^2 v - 75a_0^4 u^2 v/4 + \cdots,
\end{aligned}$$

$$\begin{aligned}
9u\omega_3^2 = a_0^2/4 &+ 5a_0^4/16 + u + 3a_0^2 u/4 + 5a_0^4 u/16 - 9a_0^2 u^2/4 - \\
&5a_0^4 u^2/2 + 2a_0^2 u^3 + 25a_0^4 u^3/4 - 125a_0^4 u^4/16 + \\
&65a_0^4 u^5/16 - 5a_0^4 v/16 - 3a_0^2 uv/2 - 5a_0^4 uv/2 + \\
&3a_0^2 u^2 v/2 + 75a_0^4 u^2 v/8 + \cdots,
\end{aligned} \tag{21}$$

$$\begin{aligned}
25v\omega_3^2 = a_0^4/16 &+ 3a_0^2 u/4 + 15a_0^4 u/16 - 3a_0^2 u^2/4 - 5a_0^4 u^2/2 + \\
&25a_0^4 u^3/8 - 25a_0^4 u^4/16 - a_0^4 u^5/16 + v + 3a_0^2 v/2 + \\
&25a_0^4 v/16 - 9a_0^2 uv/2 - 35a_0^4 uv/4 + 15a_0^2 u^2 v/4 \\
&+75a_0^4 u^2 v/8 + \cdots.
\end{aligned}$$

From the first equation of Eq. (21),

$$\begin{aligned}
\omega_3^2 = &(1 + 3a_0^2/4 + 5a_0^4/8 - u - 3a_0^2 u/2 - 25a_0^4 u/16 + 9a_0^2 u^2/4 + \\
&15a_0^4 u^2/4 - 3a_0^2 u^3/2 - 25a_0^4 u^3/4 + 25a_0^4 u^4/4 - 45a_0^4 u^5/16 - \\
&v - 9a_0^2 v/4 - 45a_0^4 v/16 + 9a_0^2 uv/2 + 10a_0^4 uv - \\
&3a_0^2 u^2 v - 75a_0^4 u^2 v/4 + \cdots)/(1-u-v).
\end{aligned} \tag{22}$$

Eliminating ω_3^2 from last two equations of Eq. (21) with the help of Eq. (22) and the simplified form of nonlinear algebraic equations of u and v are as follows

$$\begin{aligned}
u = \lambda_0 \big(&4 + 5a_0^2 + 168u^2 - 256u^3 - 4v - 10a_0^2 v + \\
&288uv + 128uv/a_0^2 - 564u^2 v + \cdots\big)
\end{aligned} \tag{23}$$

$$v = \mu_0 \left(a_0^2 + 12u + 14a_0^2 u - 24u^2 + 12u^3 + 492uv + 384uv/a_0^2 - 756u^2 v + 540u^3 v + \cdots\right) \tag{24}$$

where λ_0 is defined as Eq. (16) and $\mu_0 = a_0^2/(384 + 276a_0^2 + 226a_0^4)$.
The algebraic relation between λ_0 and μ_0 is

$$\mu_0 = \frac{(64 + 50a_0^2 + 45a_0^4)\lambda_0}{192 + 138a_0^2 + 113a_0^4}. \tag{25}$$

Therefore, Eq. (24) takes the form

$$v = \frac{(64 + 50a_0^2 + 45a_0^4)\lambda_0}{192 + 138a_0^2 + 113a_0^4}(a_0^2 + 12u + 14a_0^2 u - 24u^2 + 12u^3 + 492uv + 384uv/a_0^2 - 756u^2 v + 540u^3 v + \cdots). \tag{26}$$

The power series solutions of Eq. (23) and Eq. (26) are obtained in terms of λ_0

$$u = (4 + 5a_0^2)\lambda_0 - \frac{(256 + 840a_0^2 + 680a_0^4 + 450a_0^6)}{192 + 138a_0^2 + 113a_0^4} a_0^2 \lambda_0^2 + \cdots, \tag{27}$$

$$v = \left(\frac{(64 + 50a_0^2 + 45a_0^4)a_0^2\lambda_0^2 + (3072 + 9824a_0^2)\lambda_0^2}{192 + 138a_0^2 + 113a_0^4} + \frac{(12440a_0^4 + 8720a_0^6 + 3150a_0^8)\lambda_0^2}{192 + 138a_0^2 + 113a_0^4} + \cdots\right). \tag{28}$$

Now substituting the values of u and v from Eqs. (27)-(28) into Eq. (22), the third-order approximate angular frequency is

$$\omega_3 = \left[-\left(\frac{-1536 + 2352a_0^2 + 12908a_0^4}{8(192 + 138a_0^2 + 113a_0^4)} + \frac{18544a_0^6 + 12805a_0^8 + 5025a_0^{10}}{8(192 + 138a_0^2 + 113a_0^4)}\right)\lambda_0 + \cdots\right]^{\frac{1}{2}}. \tag{29}$$

Therefore, a third-order approximation periodic solution of Eq. (11) is defined as Eq. (20) where u, v and ω_3 are respectively given by the Eqs. (27)-(28) and Eq. (29).

5. Results and Discussion

We illustrate the accuracy of the second and third order approximate frequencies and their relative errors $Er(\%)$ obtained in present study by using HBM of the cubic-quintic Duffing oscillator comparing with previously obtained and the exact frequency ω_{ex} are the following Table 1. For this nonlinear problem, the exact frequency is stated by S.K. Lai et al. (Lai, 2008). It can clearly be seen that all the approximate frequencies obtained in this article applying

Table 1: Comparison of the approximate frequencies obtained by our technique and other existing result with exact frequency ω_{ex} (Ganji, 2009) of cubic-quintic Duffing oscillator.

a_0	ω_2 $Er(\%)$	ω_3 $Er(\%)$	ω_{ex}	$\omega^{EBM}_{(Ganji,\ 2009)}$ $Er(\%)$
0.1	1.003772	1.003772	1.003770	1.003773
	0.0001	*0.0001*		*0.0002*
0.3	1.035540	1.035538	1.035540	1.035492
	0.0000	*0.0001*		*0.0046*
0.5	1.106591	1.106545	1.106540	1.106356
	0.0046	*0.0004*		*0.0166*
1.0	1.526041	1.523762	1.523590	1.527720
	0.1608	*0.0112*		*0.2710*
3.0	7.367509	7.284161	7.268630	7.417768
	1.3603	*0.2136*		*2.0518*
5.0	19.482606	19.231700	19.181500	19.608793
	1.5697	*0.2617*		*2.2276*
8.0	49.086138	48.428975	48.294600	49.390624
	1.6389	*0.2782*		*2.2694*
10.0	76.421355	75.389512	75.177400	76.889585
	1.6546	*0.2821*		*2.2775*

HBM is better than those obtained previously by D. D Ganji et al. (Ganji, 2009), S. S. Ganji et al. (Ganji, 2012a) and S.K. Lai et al. (Lai, 2008). In adding, the third-order approximate results give almost similar comparing with exact frequencies. It has been mentioned that the procedure of D. D Ganji et al. (Ganji, 2009), S. S. Ganji et al. (Ganji, 2012a) and S.K. Lai et al. (Lai, 2008) is laborious especially for obtaining higher order approximations. Comparing the results obtained in this article with previously obtained by several authors it has been shown that the proposed method is simpler than several exiting procedures. The advantages of this method include its simplicity and computational efficiency, and the ability to objectively find better agreement in third-order approximate frequency.

ω_2 and ω_3 respectively denote second- and third- order approximate frequencies obtained by HBM. ω_{ex} represents the exact frequency obtained by numerical method. $\omega_{[15]}^{EBM}$ represents the approximate frequency obtained in (Ganji, 2009). $Er(\%)$ denotes the percentage error obtained by the relation $|\frac{\omega_{ex}-\omega_i}{\omega_{ex}} \times 100|$ where $i = 2, 3$.

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